Doughnuts and the Euler Characteristic

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BDI

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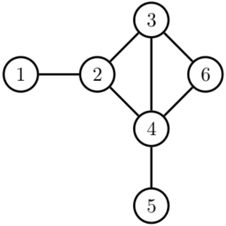
Mathematical Disclaimer: Some definitions can be hard to get ones head around so if at any time you want me to remind you of a certain definition or if you have any question no matter how silly, let me know!

Surfaces and Spaces

Examples of Spaces

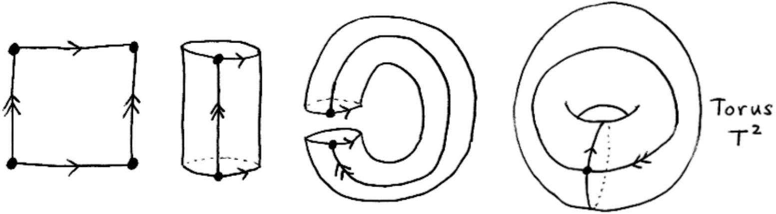
- The real line: $\mathbb{R}$, the plane: $\mathbb{R}^2$, general euclidean spaces: $\mathbb{R}^n$.
- A single point. $\mathbb{R}^0$!
- A graph. Eg:



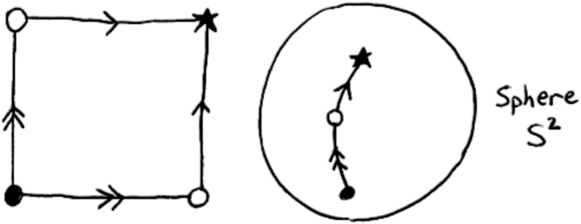
- More generally any shape, eg in 2D or 3D.
- Disc: $\mathbb{D}_n = \left\{x \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 \leq 1\right\}$.
- Sphere: $S_n = \left\{x \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 = 1\right\}$.

Examples of Surfaces

- The doughnut/torus



- The Sphere



Surfaces

So what is a surface: well. Certainly a line is not a surface. Nor is a solid sphere as it has an inside.

Definition

We define a space to be a **surface** if it is locally flat. So that if we look at a really small point of it it seems flat.

The Earth is an example as its just a sphere:

So is a plane. (Ie a tabletop.)

Homeomorphisms

To make the previous definition more precise we need to define the concept of Homeomorphism between spaces.

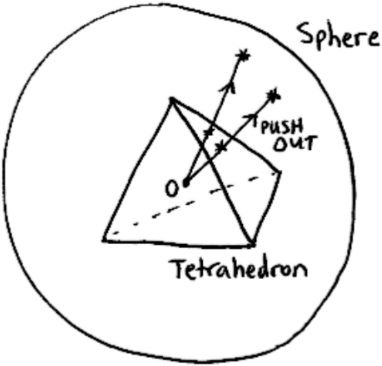
Definition

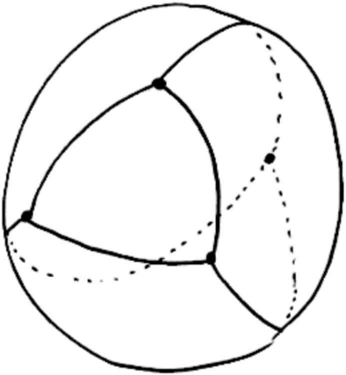
Given 2 spaces S and T, we say that S and T are homeomorphic if there exists a function $f : S \longrightarrow T$ such that f is a bijection and f is continuous with continuous inverse.

Note that if $f : R \longrightarrow S$ and $g : S \longrightarrow T$ are homeomorphisms, then clearly $g \circ f$ is a homeomorphism.

Examples of Homeomorphisms

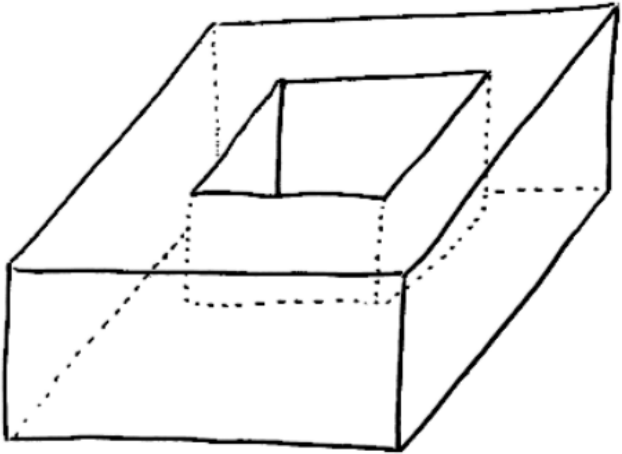
- The empty square and the circle.
- The square and the disc.
- The pyramid and the sphere.





Examples of Homeomorphisms continued

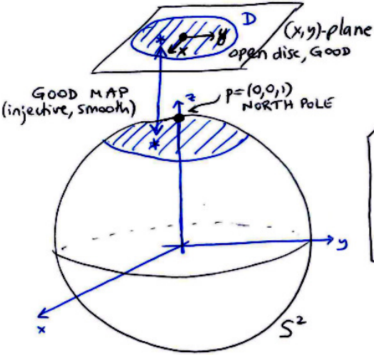
- The Torus and a rectangular torus.



Formal definition of a surface

Definition

A general space S is a surface if the neighbourhood of each point p is locally homeomorphic to the disc:
$\mathbb{D}_2 = \{x = (x_0, x_1, x_2) \in \mathbb{R}^2 : \sum x_i^2 \leq 1\}$



Other examples: eg the torus, the plane, any surface of a 3D shape.

The Euler Characteristic

Cellular Decomposition

Let $\mathbb{D}_n = \left\{x \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i^2 \leq 1\right\}$ be the n-disc. (Go through examples.)

Definition

Given a space S, an n-**cell** is a continuous map $f : \mathbb{D}_n \longrightarrow S$.

Definition

Given a space S, a **cellular decomposition** is a finite collection of maps $\{f_1, f_2, \dots\}$ such that each f_i is an n-cell for some integer n and

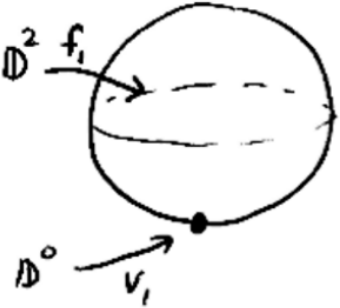
- For each i, $f_i : \mathrm{Int}(\mathbb{D}_n) \longrightarrow S$ is a homeomorphism onto its image.
- the boundary of the disc is mapped into the image of lower dimensional cells (except for zero cells)
- S is partitioned by the interiors of the cells.

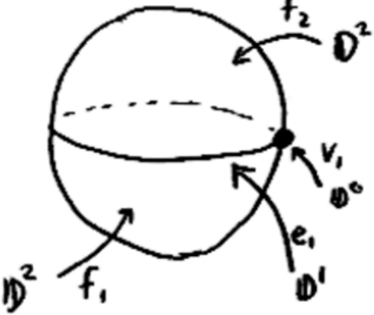
Ie it is a set of maps sending points, discs, solid spheres etc to the space S. Don't worry, we're going to see a lot of examples of these. Please ask questions here.

Examples of Cellular Decompositions

- A point.
- A line between two points.
- A graph.
- A graph with faces.
- A circle.
- The Sphere.
- The Torus.

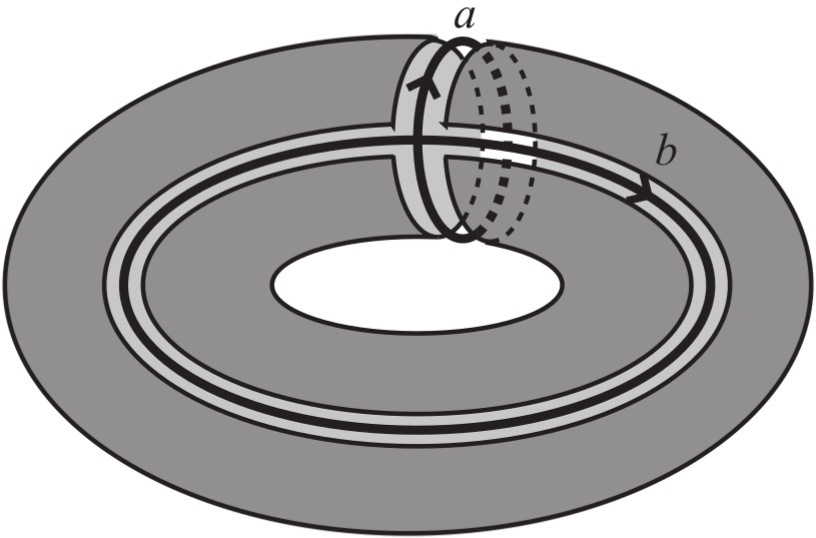
Cellular Decompositions of the Sphere (multiple)





What about the solid sphere? Ie D_4?

Cellular Decomposition of the Torus



The Euler Characteristic

Any compact space S has a cellular decomposition and the sum:

$$\chi(S) = (\#0\text{-cells}) - (\#1\text{-cells}) + (\#2\text{-cells}) - (\#3\text{-cells}) + \dots$$

is the same for any cellular decomposition. It is called the **Euler Characteristic of S**.

Proof.

No clue. Its hard to show. □

In particular if $\{f_1, f_2, \dots\}$ is a cellular decomposition of S such that f_i is a homeomorphism $f_i : \mathbb{D}^{n_i} \longrightarrow S$, then

$$\chi(S) = \sum_i (-1)^{n_i}.$$

Examples of Euler Characteristics

- A point: 1.
- A line between two points: 2 - 1 = 1. (Note can't eg just use a line as you need to clue the endpoints onto exists lower dimensional cells.)
- A circle: 1 - 1 = 0.
- The sphere: 1 - 0 + 1 = 2. Also, 1 - 1 + 2 = 2 for the other decomposition.
- The solid sphere: 1 - 0 + 1 - 1 = 1.
- The Torus: 1 - 2 + 1 = 0.
- The Solid Torus: 1 - 2 + 2 - 1 = 0.

Theorem

If M, N are homeomorphic compact spaces then $\chi(M) = \chi(N)$.

Proof.

Suppose that $g : M \longrightarrow N$ is a homeomorphism (ie g is bijective, continuous and has continuous inverse). Suppose that $\{f_1, f_2, \ldots\}$ is a cellular decomposition of M such that f_i is a homeomorphism

$$f_i : \mathbb{D}^{n_i} \longrightarrow M.$$

Then $\{g \circ f_1, g \circ f_2, \ldots\}$ is a cellular decomposition of M as

$$g \circ f_i : \mathbb{D}^{n_i} \longrightarrow N$$

is a homeomorphism as it is a composition of homeomorphisms. So

$$\chi(N) = \sum_i (-1)^{n_i} = \chi(M).$$

□

The Euler Characteristic of a Planar Graph

- Recall a planar graph is a graph that can be drawn in the plane.
- By taking the 0-cells to be the vertices, the 1-cells to be the edges and the 2-cells to be the faces it is clear that a planar graph has euler characteristic $V - E + F$.
- What isn't clear is that this is the same for all connected planar graphs as the following Theorem states:

Theorem

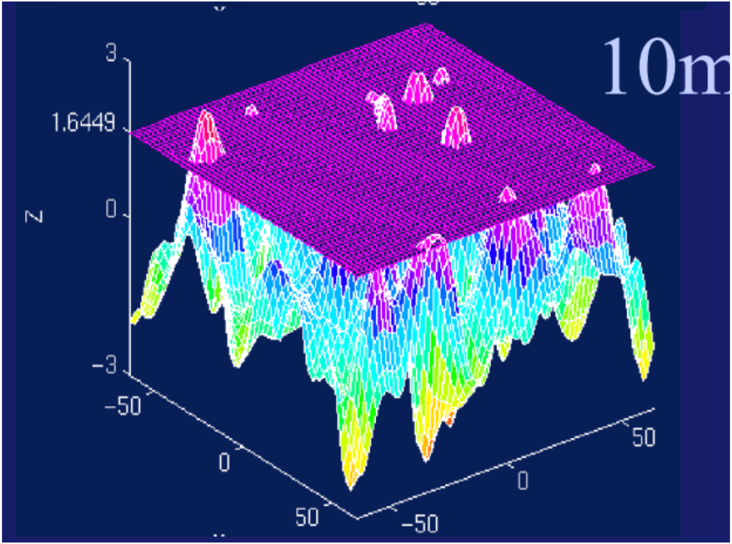
Given a connected planar graph $\mathcal{G}$, $\chi(\mathcal{G}) = 2$ ie $V - E + F = 2$.

Proof.

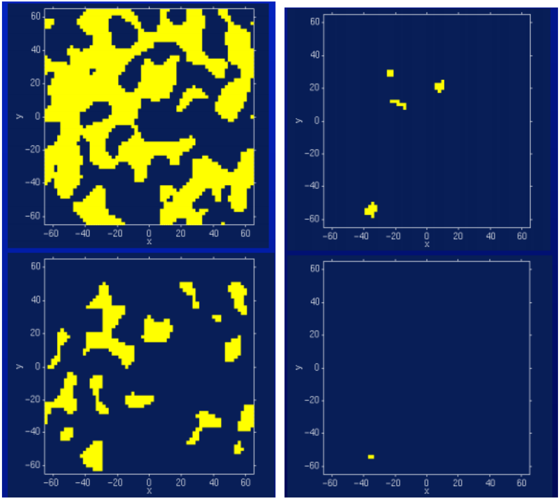
By induction. A graph with one vertex and E edges (which are all then loops) has $E + 1$ faces. And indeed $V - E + F = 1 - E + E + 1 = 2$! For a graph with n vertices, choose an edge e connecting two different vertices, and contract it. This decreases both the number of vertices and edges by one, and the result then holds by induction. □

The Euler Characteristic in Neuroimaging

Illustration in 2D



At High Thresholds, Euler Char is the number of Maxima



Using the Euler Characteristic

Let $W : \Theta \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a random field and let $\mathcal{A}_u$ be the excursion set when the threshold is u ie

$$\mathcal{A}_u = \{\theta \in \Theta : W(\theta) \geq u\}.$$

Using the Euler Characteristic

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For high thresholds u, the number of maxima above the threshold is $M_u = \chi(\mathcal{A}_u)$. So

$$\mathbb{E}[M_u] = \mathbb{E}[\chi(\mathcal{A}_u)].$$

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So

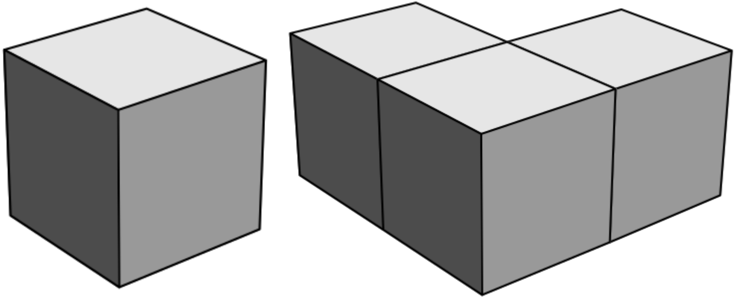
$$\mathbb{P}\left(\sup_{\theta \in \Theta} W(\theta) > u\right) = \mathbb{P}(M_u \geq 1) \leq \mathbb{E}[M_u] \approx \mathbb{E}[\chi(\mathcal{A}_u)].$$

Doughnuts


PureWow

Euler Characteristic in 3D

Euler Characteristic on a 3D Lattice

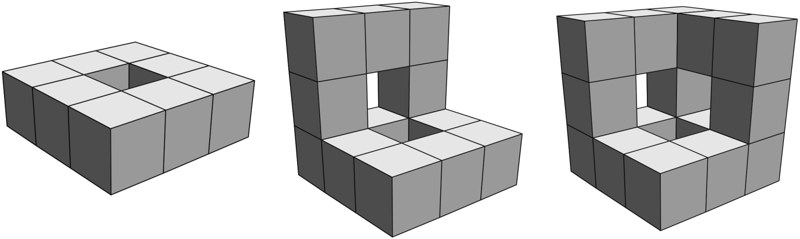


What are Euler Characteristics? Well we have easy cellular decompositions as vertices edges, faces and the insides. The number of insides is the number of boxes. So the euler characteristic is:

$$\chi = V - E + F - B$$

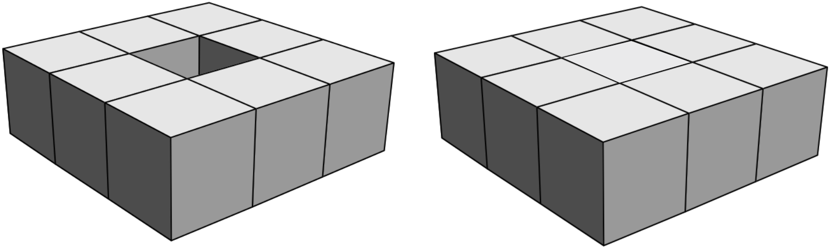
Notice that: 8-12+6-1=1 and 16-28+16-3=1. In general as homeomorphic to a sphere a solid lattice has Euler Chactacteristic 1.

Contribution of Holes



What are Euler Characteristics? Well: 32-64+40-8=0 (as homeomorphic to the torus!), 48-100+64-13=-1 and 56-120+78-16=-2 so for each additional hole the EC decreases by 1.

Contribution of Holes



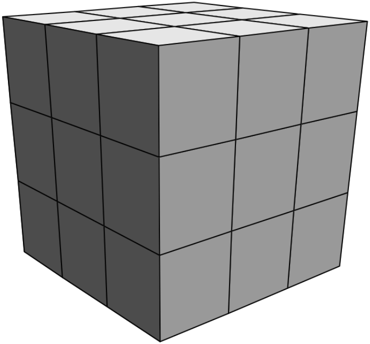
Is there an easy way to see that euler characteristic $\chi(T)$ of the rectangular torus T is 0? Well, if we add in two faces and an inside we get the shape on the right hand side which has Euler Characteristic 1. So it follows that:

$$\chi(T) + 2 - 1 = 1 \implies \chi(T) = 0.$$

Repeating this argument for each hole in the lattice it follows that each hole contributes -1 to the Euler Characteristic.

Contribution of Hollows

How about the following tennis ball shape (where the centre cube has been taken out)?



Well 64-144+108-26 = 2. The inside hollow can be filled by a 3-cell which would then result in a connected lattice. As such:

$$\chi(\text{Tennis Ball}) - 1 = 1 \implies \chi(\text{Tennis Ball}) = 2$$

EC on a 3D Cubic Lattice

So we have (more or less) proven the following Theorem:

Theorem

The Euler Characteristic of a cubic lattice is:

$$\#(Connected\ components) - \#(Holes) + \#(Hollows)$$

Why more or less? Well...

Adler's Result

We want to prove the following:

Theorem 5.3.1

Let $X(\mathbf{t})$ be a zero-mean, homogeneous Gaussian random field on $\mathscr{R}^N$ and S a subset of $\mathscr{R}^N$, and let both satisfy the conditions of Theorem 5.2.2. Then the mean value of the DT characteristic of the excursion set $A = A_u(X, S)$ is given by

$$(5.3.10)\qquad E\{\chi(A)\} = \lambda(S)(2\pi)^{-(N+1)/2}|\Lambda|^{1/2}\sigma^{-(2N-1)}\exp\left(-\frac{u^2}{2\sigma^2}\right) \times \sum_{j=0}^{[(N-1)/2]}(-1)^j a_j \sigma^{2j}\binom{N-1}{2j}u^{N-1-2j}.$$

Morse's Theorem

Theorem

Let $F : S \subset \mathbb{R}^N \longrightarrow \mathbb{R}$ be a (suitably nice) function for some compact subset S. Let $A_u = \{t \in S : F(t) \geq u\}$, then

$$\chi(A_u) = (-1)^{N-1} \sum_{k=0}^{N-1} (-1)^k m_k$$

where m_k is the number of points $t \in S$ such that:

- $F(t) = u$
- $F_j := \frac{\partial F}{\partial t_j}(t) = 0$ *for* $j = 1, 2, \ldots, N-1$
- $F_N := \frac{\partial F}{\partial t_N}(t) > 0$
- *The* $N-1 \times N-1$ *matrix:* $D(t) = (F_{i,j}(t))_{i,j=1,\ldots,N-1} = (\frac{\partial^2 F}{\partial t_j t_i}(t))_{i,j=1,\ldots,N-1}$ *has* k *negative eigenvalues.*

Morse's Theorem

Theorem

Let $X : S \subset \mathbb{R}^N \longrightarrow \mathbb{R}$ *be a (suitably nice) random field for some compact subset* S. *Let* $A_u = \{t \in S : X(t) \geq u\}$, *then*

$$\mathbb{E}[\chi(A_u)] = (-1)^{N-1} \sum_{k=0}^{N-1} (-1)^k \mathbb{E}[m_k]$$

where m_k *is the number of points* $t \in S$ *such that:*

- $X(t) = u$
- $X_j := \frac{\partial X}{\partial t_j}(t) = 0$ *for* $j = 1, 2, \ldots, N-1$
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Given a random field $X : \mathbb{R}^N \longrightarrow \mathbb{R}$, and $u \in \mathbb{R}$, define

$$f(t) = [X(t) - u, X_1(t), \ldots, X_{N-1}(t)]$$

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$$f(t) = [X(t) - u, X_1(t), \dots, X_{N-1}(t)]$$

and

$$g(t) = [X_N(t), X_{11}(t), \dots, X_{N-1,N-1}(t)]$$

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For $k = 0, \ldots, N-1$, let

$$A_k = \{g(t) : X_N(t) > 0 \text{ and } D(t) \text{ has } k \text{ negative eigenvalues}\}.$$

Then for $k = 0, \ldots, N-1$, m_k is the number of points $t \in S$ such that

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We want to find $\mathbb{E}[m_k]$ so it is sufficient to, given vector random fields, $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $g : \mathbb{R}^N \longrightarrow \mathbb{R}^k$ and some set $A \subset \mathbb{R}^k$, find an expression for the expected number of points $t \in S$ such that

$$f(t) = 0 \text{ and } g(t) \in A.$$

Kac-Rice Formula

Let δ_ϵ be constant on the ball $B_\epsilon(0) = \left\{t \in \mathbb{R}^N : \|t\| < \epsilon\right\}$ and zero elsewhere and normalized so that:

$$\int_{B_\epsilon(0)} \delta_\epsilon(t)\, dt = 1.$$

Theorem

Let $f : S \subset \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $g : S \subset \mathbb{R}^N \longrightarrow \mathbb{R}^K$ be continuous functions on a compact set $S \subset \mathbb{R}^N$ and let $A \subset \mathbb{R}^k$ be open. Let M be the number of points for which

$$f(t) = 0 \in \mathbb{R}^N \text{ and } g(t) \in A \subset \mathbb{R}^K$$

Then:

$$M = \lim_{\epsilon \longrightarrow 0} \int_S \delta_\epsilon(f(t)) \mathbb{1}_A(g(t)) |\det(\nabla f(t))|$$

Proof.

Let $N = K = 1$. Then for small enough η, ϵ, we have:

$$B_\epsilon(0) \subset f|_{B_\eta(t^*)}(B_\eta(t^*))$$

so $f|_{B_\eta(t^*)} : f|^{-1}_{B_\eta(t^*)}(B_\epsilon(0)) \longrightarrow B_\epsilon(0)$ is 1-1. We have:

$$\int_{B_\epsilon(0)} \delta_\epsilon(t)\, dt = 1.$$

and so a change of variables yields

$$\int_{f|^{-1}_{B_\eta(t^*)}(B_\epsilon(0))} \delta_\epsilon(f(t))|\det \nabla f(t)|\, dt = 1.$$

In particular this implies that for small enough ϵ:

$$M = \int_S \delta_\epsilon(f(t)) \mathbb{1}_A(g(t)) |\det(\nabla f(t))|\, dt$$

Kac-Rice formula

If we now let f and g be random fields then

$$M = \lim_{\epsilon \longrightarrow 0} \int_S \delta_\epsilon(f(t)) \mathbb{1}_A(g(t)) |\det(\nabla f(t))| \, dt$$

is random, in particular (under suitable assumptions) we have:

$$\begin{aligned}
\mathbb{E}[M] &= \int_S \int_{\mathbb{R}^{N(N+1)/2}} \int_{\mathbb{R}^K} \mathbb{1}_A(v) |\det(\nabla y)| \lim_{\epsilon \longrightarrow 0} \int_{\mathbb{R}^N} \delta_\epsilon(x) p_t(x, \nabla y, v) dx d\nabla y \\
&= \int_S \int_{\mathbb{R}^{N(N+1)/2}} \int_{\mathbb{R}^K} \mathbb{1}_A(v) |\det(\nabla y)| p_t(0, \nabla y, v) \, d\nabla y \, dv \, dt \\
&= \int_S \mathbb{E}[|\det(\nabla y)| \mathbb{1}_A(g(t)) | f(t) = 0] p_t(0) \, dt.
\end{aligned}$$

Bibliography